

2D Transformations

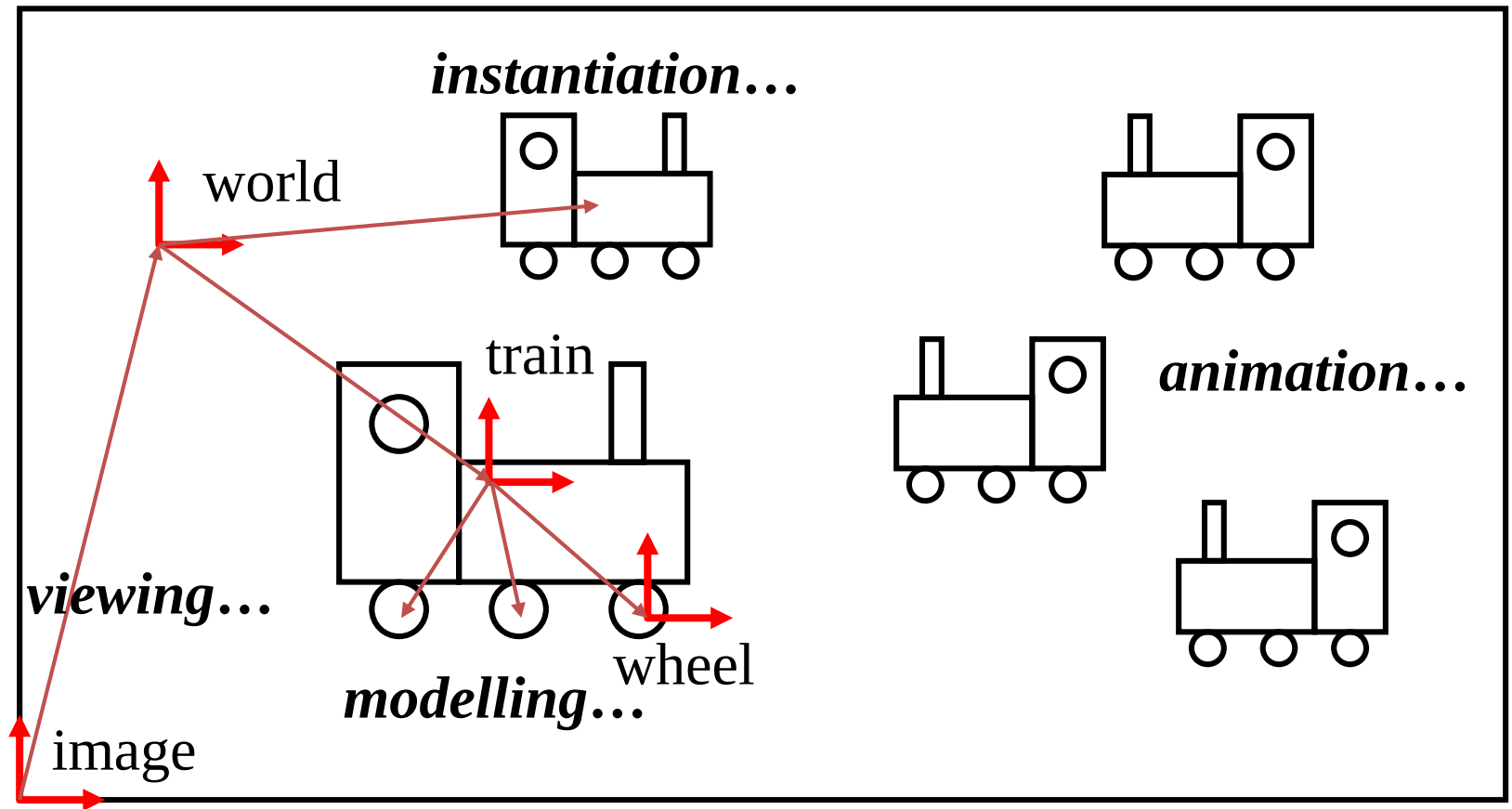
Computer Graphics

Lecture 2

Overview

- Why transformations?
- Basic transformations:
 - translation, rotation, scaling
- Combining transformations
 - homogenous coordinates, transform. Matrices
- First 2D, next 3D

Transformations



Why transformation?

- Model of objects

world coordinates: *km, mm, etc.*

Hierarchical models::

human = torso + arm + arm + head + leg + leg

arm = upperarm + lowerarm + hand ...

- Viewing

zoom in, move drawing, etc.

- Animation

linear algebra review

- matrices
 - notation
 - basic operations
 - matrix-vector multiplication

matrices

- notation for matrices and vectors
 - use column form for vectors

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = [m_{ij}] \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [v_1 \quad v_2]^T$$

matrix operations

- addition

$$T = M + N \Rightarrow [t_{ij}] = [m_{ij} + n_{ij}]$$

- scalar multiply

$$T = \alpha M \Rightarrow [t_{ij}] = [\alpha m_{ij}]$$

matrix operations

- matrix-matrix multiply
 - row-column multiplication
 - not commutative
 - associative

$$T = MN \Rightarrow [t_{ij}] = \left[\sum_k m_{ik} n_{kj} \right]$$

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$$

matrix operations

- matrix-vector multiply
 - row-column multiplication

$$\mathbf{u} = M\mathbf{v} \Rightarrow [u_i] = \left[\sum_k m_{ik} v_k \right]$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

special matrices

- identity

$$I = [i_{ij}] = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$MI = IM = M$$

- zero

$$O = [o_{ij}] = 0$$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M + O = M$$

matrix operations

- transpose
 - flip along the diagonal

$$T = M^T \Rightarrow [t_{ij}] = [m_{ji}]$$

- inverse
 - will not compute explicitly in this course

$$T = M^{-1} \Rightarrow TM = MM^{-1} = M^{-1}M = I$$

matrix operations properties

- linearity of multiplication

$$\alpha(A + B) = \alpha A + \alpha B$$

$$M(\alpha A + \beta B) = \alpha MA + \beta MB$$

- associativity of multiplication

$$A(BC) = (AB)C$$

- transpose and inverse of matrix multiply

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

geometric transformation

- function that maps points to points

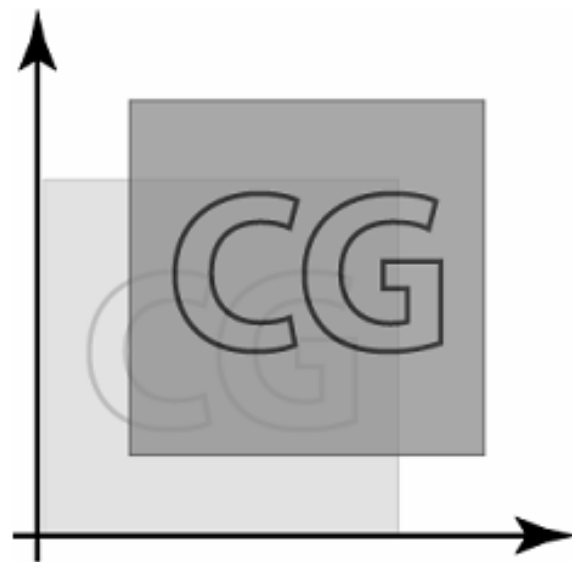
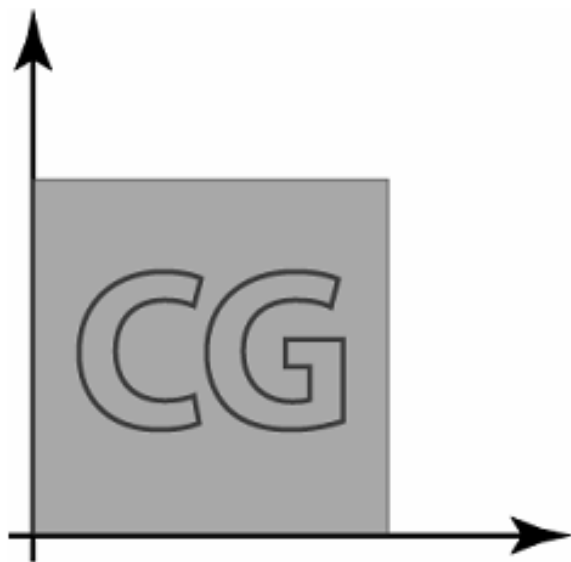
$$\mathbf{p} \rightarrow \mathbf{p}' = X(\mathbf{p})$$

- different transformations have restriction on the form of X
 - we will look at linear, affine and projections

2D transformations

translation

- simplest form $T_t(\mathbf{p}) = \mathbf{p} + \mathbf{t}$
- inverse $T_t^{-1}(\mathbf{p}) = T_{-\mathbf{t}}(\mathbf{p}) = \mathbf{p} - \mathbf{t}$



Ex. Translate a polygon with coordinates A(2,5), B(7,10) and C(10,2) by 3 units in x direction and 4 units in y direction.

Solution-

$$A' = A + T$$

$$\Rightarrow \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$B' = B + T$$

$$\Rightarrow \begin{bmatrix} 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$C' = C + T$$

$$\Rightarrow \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

linear transformation

- fundamental property

$$X(\alpha \mathbf{p} + \beta \mathbf{q}) = \alpha X(\mathbf{p}) + \beta X(\mathbf{q})$$

- can be represented in matrix form

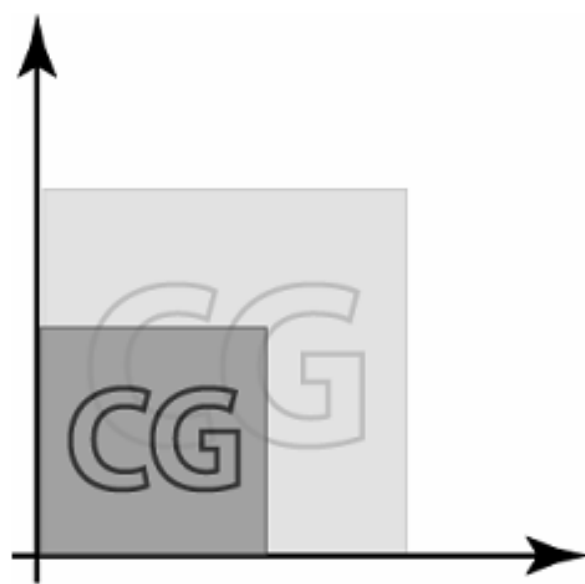
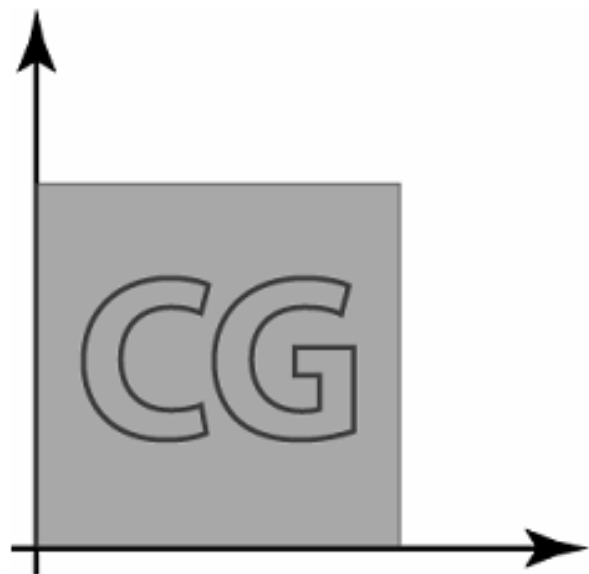
$$X(\mathbf{p}) = M\mathbf{p}$$

- other properties
 - maps origin to origin
 - maps lines to lines
 - parallel lines remain parallel
 - length ratios are preserved
 - closed under composition

uniform scale

$$S_s \mathbf{p} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} sp_x \\ sp_y \end{bmatrix}$$

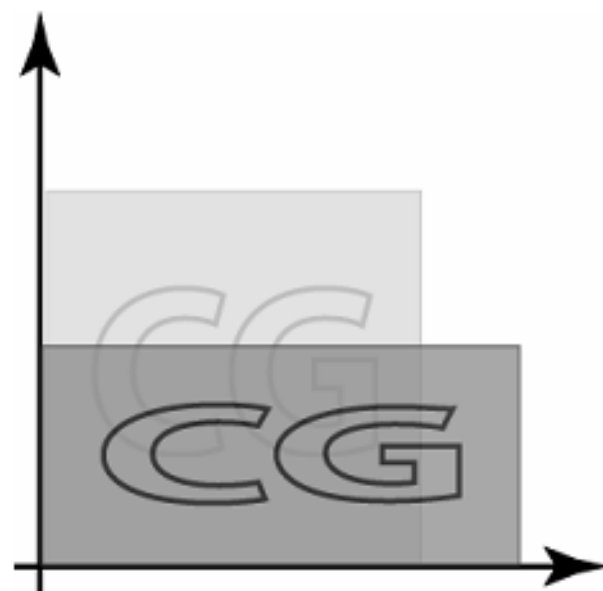
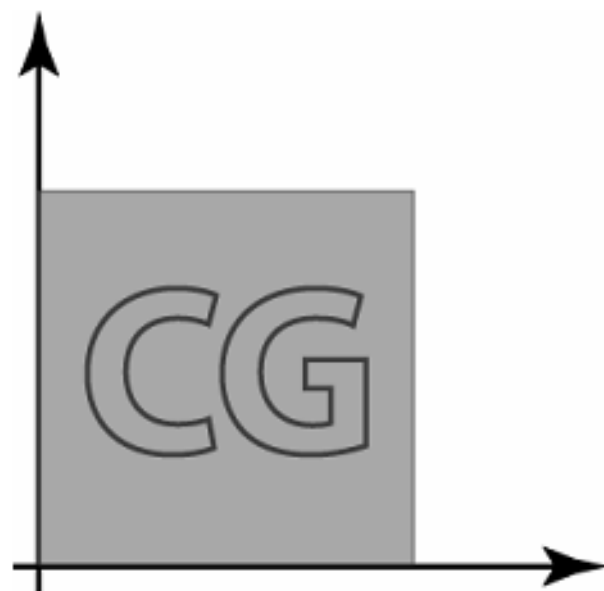
$$S_s^{-1} = S_{1/s}$$



non-uniform scale

$$S_s \mathbf{p} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} s_x p_x \\ s_y p_y \end{bmatrix}$$

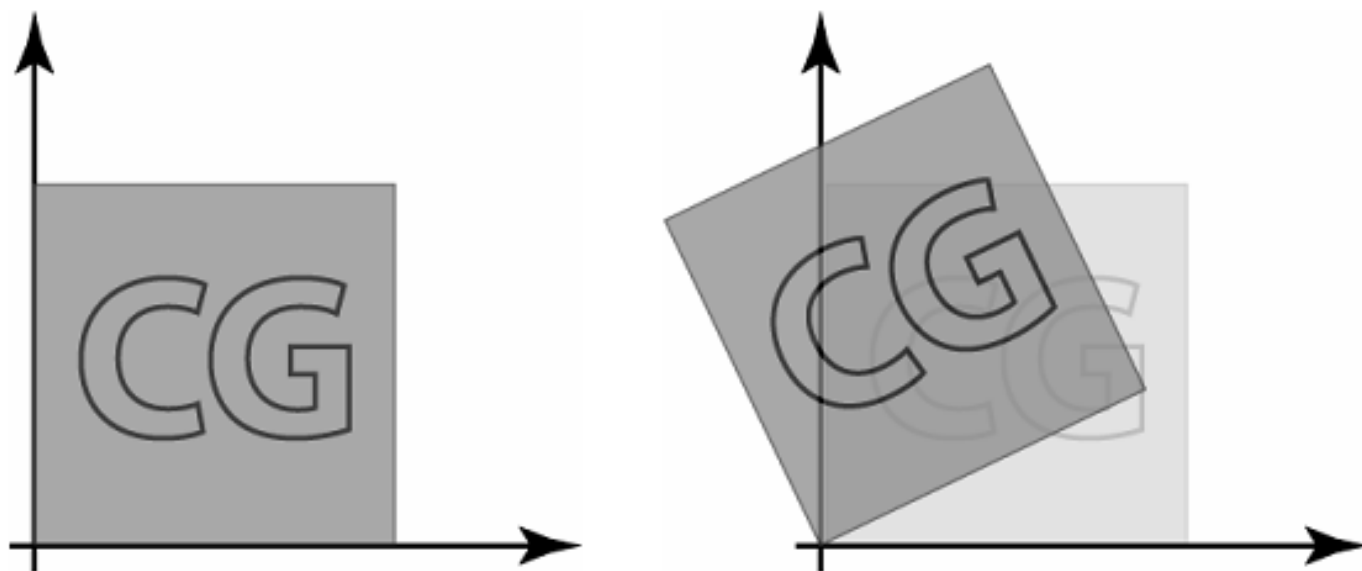
$$S_s^{-1} = S_{(1/s_x, 1/s_y)}$$



rotation

$$R_{\theta} \mathbf{p} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} p_x \cos \theta - p_y \sin \theta \\ p_x \sin \theta + p_y \cos \theta \end{bmatrix}$$

$$R_{\theta}^{-1} = R_{-\theta}$$



A point (4,3) is rotated counterclockwise by an angle of 45^0 . Find the rotation matrix and the resultant point.

Solution:

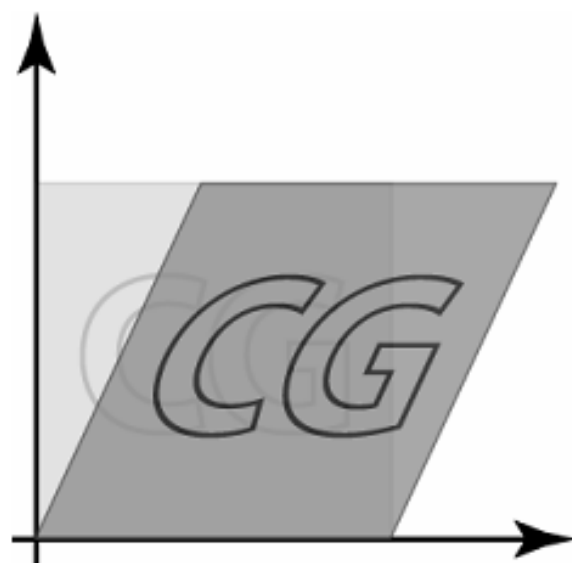
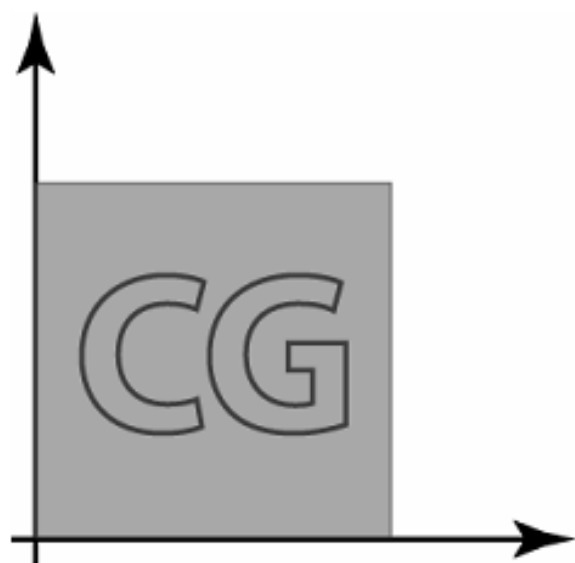
$$R_{45^o} = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix}$$

$$R_{45^o} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P' = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix}$$

shear

$$Sh_s \mathbf{p} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} p_x + s_x p_y \\ s_y p_x + p_y \end{bmatrix}$$



Shear

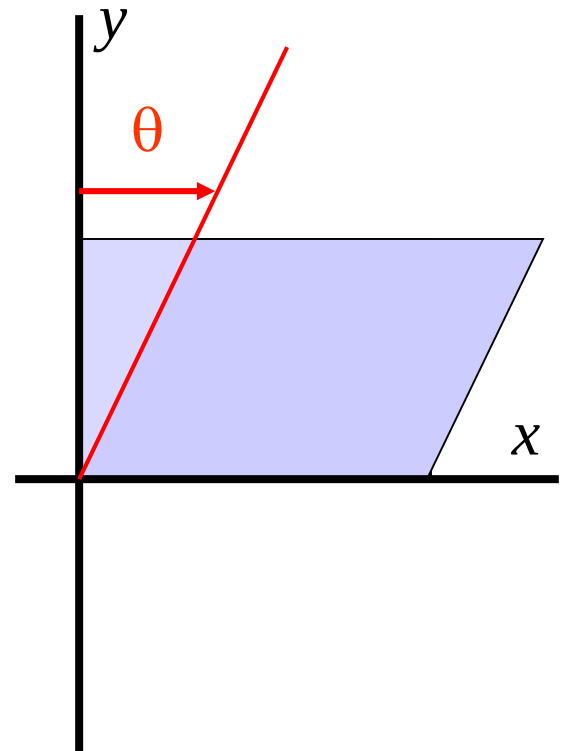
Shear the y -as:

$$x' = x + fy, \quad y' = y$$

or

$$\mathbf{P}' = \begin{pmatrix} 1 & f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{P}$$

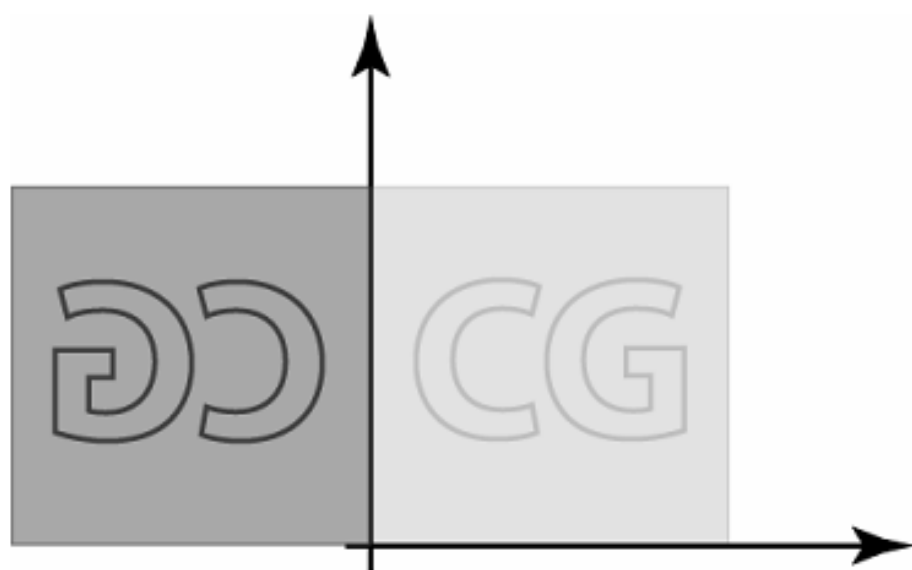
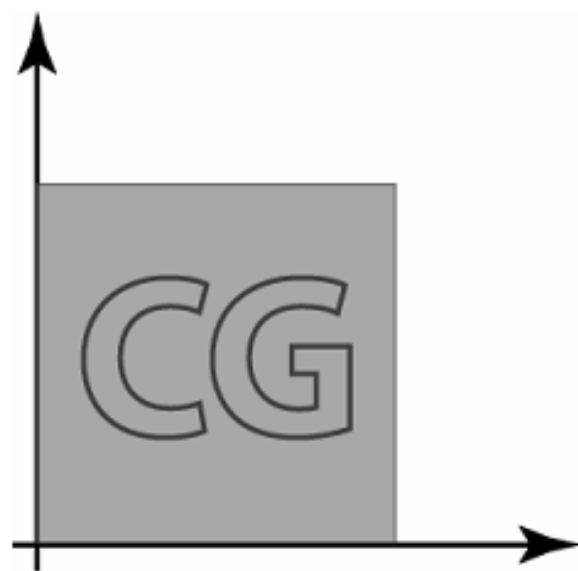
with $f = \tan \theta$



reflection

$$Rl_x \mathbf{p} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} -p_x \\ p_y \end{bmatrix} \quad Rl_y \mathbf{p} = \dots$$

$$Rl_o \mathbf{p} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} -p_x \\ -p_y \end{bmatrix}$$



combining translation and linear transforms

- represent linear together with translation
 - rigid body transformation are a subset of this

$$X_{M,t}(\mathbf{p}) = M\mathbf{p} + \mathbf{t}$$

- goal: unified format for all transformations

homogeneous coordinates

- represent points with 1 additional coordinate w
 - set it to 1 for points

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_w \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

homogeneous coordinates

- represent translation with a 3x3 matrix

$$T_t \mathbf{p} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ 1 \end{bmatrix}$$

- add one row and column to linear transforms

$$M \mathbf{p} = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11}p_x + m_{12}p_y \\ m_{21}p_x + m_{22}p_y \\ 1 \end{bmatrix}$$

affine transformations

- combining linear and translation in one matrix

$$X\mathbf{p} = M\mathbf{p} + \mathbf{t} = \begin{bmatrix} M & \mathbf{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

- properties
 - *does not map origin to origin*
 - maps lines to lines
 - parallel lines remain parallel
 - length ratios are preserved
 - closed under composition

affine transformations

translation

$$T_t = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

scale

$$S_s = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

compositing transformations

- applying one transformation after another
 - expressed by function composition

$$\mathbf{p}' = X_2(X_1(\mathbf{p})) = (X_2 \circ X_1)(\mathbf{p})$$

- for the transforms presented before,
computed by matrix multiplication

$$(X_2 \circ X_1)(\mathbf{p}) = X_2(X_1(\mathbf{p})) = M_2(M_1\mathbf{p}) = (M_2M_1)(\mathbf{p})$$

compositing transformations

- translation

$$\begin{bmatrix} I & \mathbf{t}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \mathbf{t}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{t}_1 + \mathbf{t}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

- linear transformations

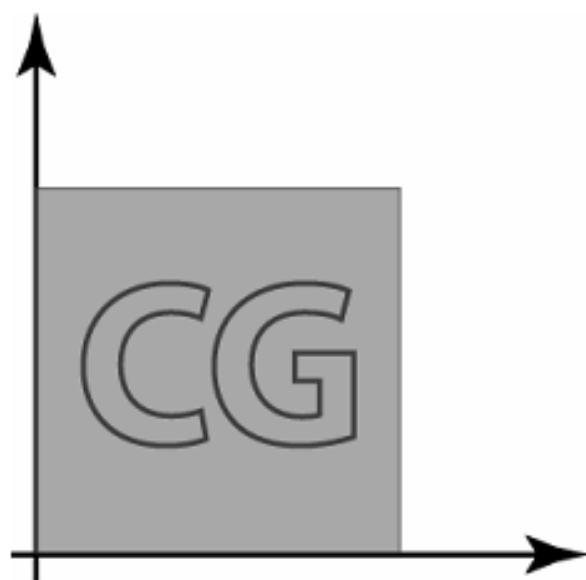
$$\begin{bmatrix} M_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} M_2 M_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

- affine transformations

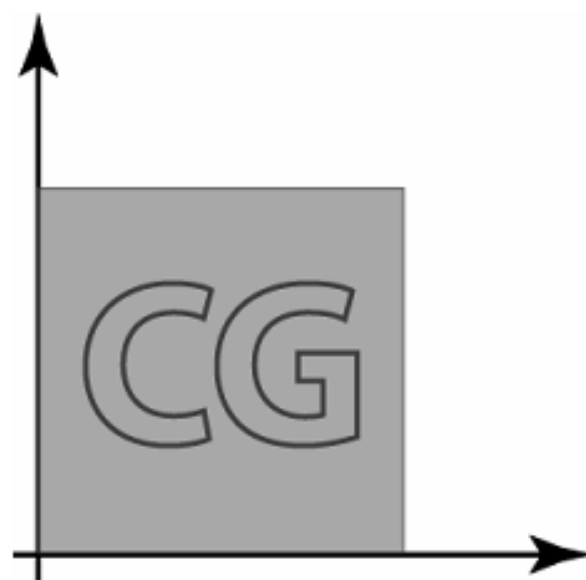
$$\begin{bmatrix} M_2 & \mathbf{t}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 & \mathbf{t}_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = \begin{bmatrix} M_2 M_1 & M_2 \mathbf{t}_1 + \mathbf{t}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

compositing transformations

- composition is not commutative



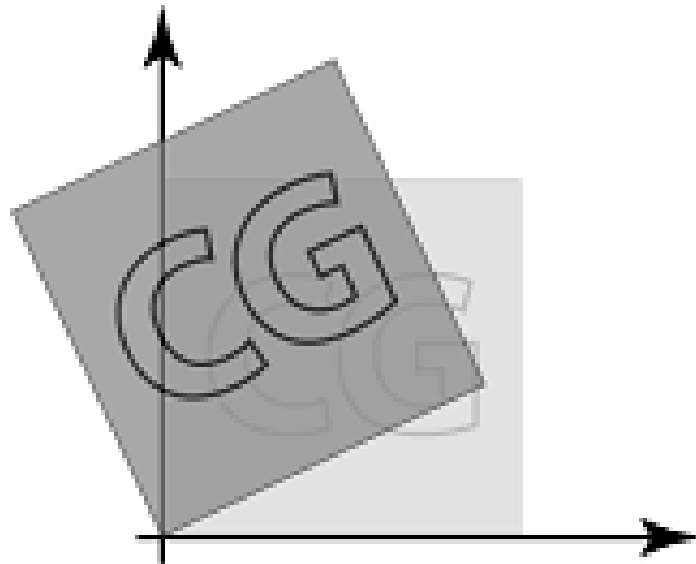
rotate,
then translate



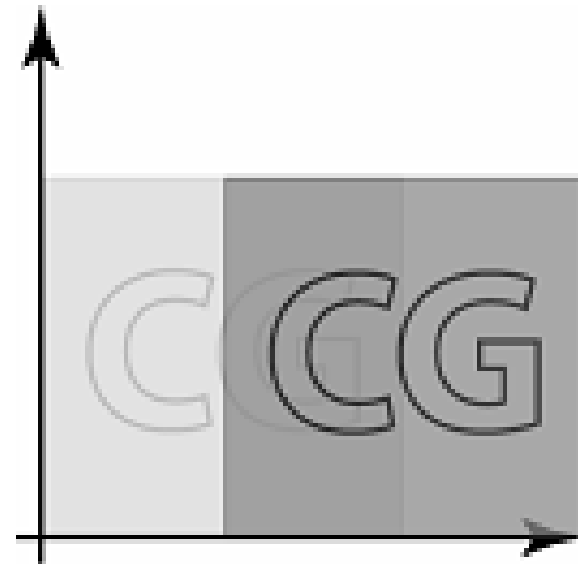
translate,
then rotate

compositing transformations

- composition is not commutative



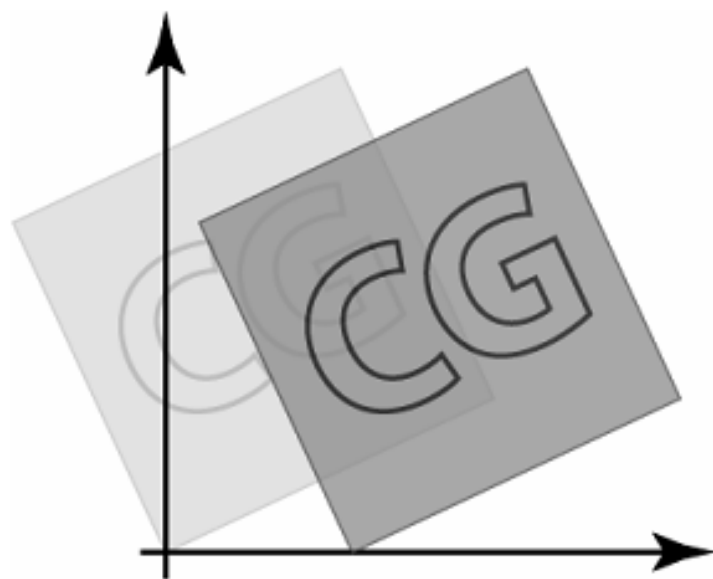
rotate,
then translate



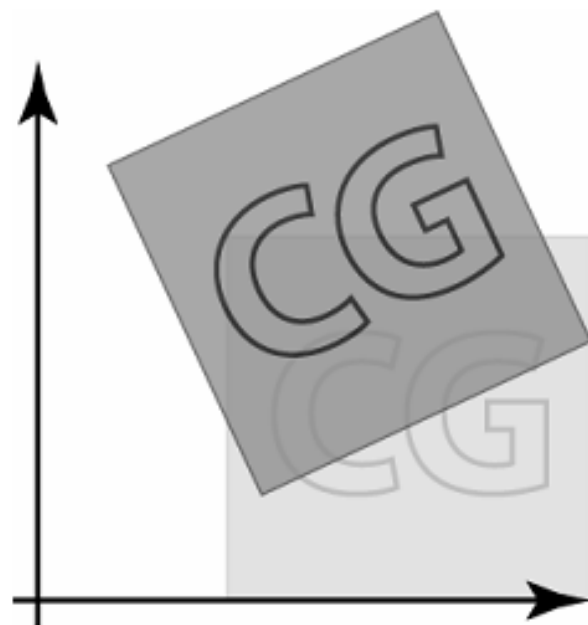
translate,
then rotate

compositing transformations

- composition is not commutative



rotate,
then translate

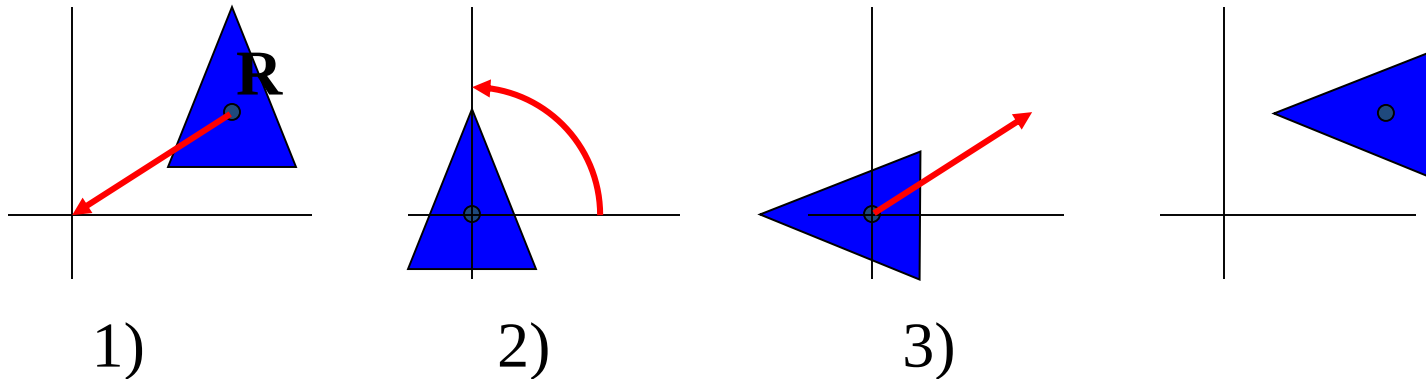


translate,
then rotate

Rotation around a point 1

Rotate over angle θ around point **R**:

- 1) Translate such that **R** coincides with origin;
- 2) Rotate over angle θ around origin;
- 3) Translate back.

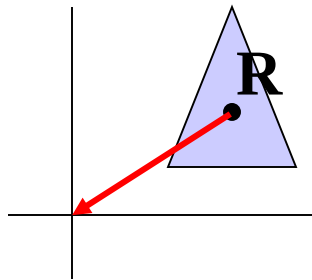


Rotation around point 2

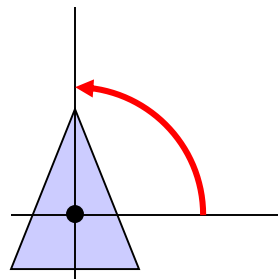
1) $\mathbf{P}' = \mathbf{T}(-R_x, -R_y)\mathbf{P}$

2) $\mathbf{P}'' = \mathbf{R}(\theta)\mathbf{P}' = \mathbf{R}(\theta)\mathbf{T}(-R_x, -R_y)\mathbf{P}$

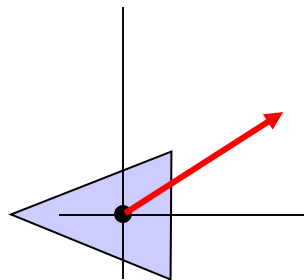
3) $\mathbf{P}''' = \mathbf{T}(R_x, R_y)\mathbf{P}''$
 $= \mathbf{T}(R_x, R_y)\mathbf{R}(\theta)\mathbf{P}'$
 $= \mathbf{T}(R_x, R_y)\mathbf{R}(\theta)\mathbf{T}(-R_x, -R_y)\mathbf{P}$



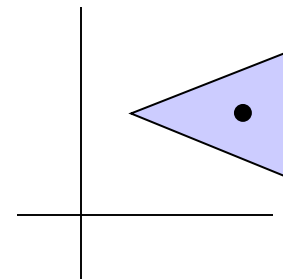
1)



2)



3)

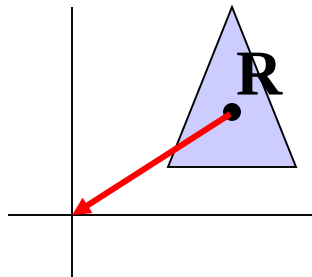


Rotation around point 2

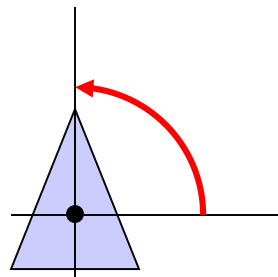
$$1-3) \mathbf{P}''' = \mathbf{T}(R_x, R_y) \mathbf{R}(\theta) \mathbf{T}(-R_x, -R_y) \mathbf{P}$$

or

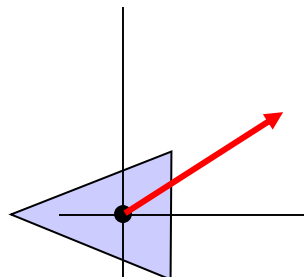
$$\mathbf{P}''' = \begin{pmatrix} \cos \theta & -\sin \theta & R_x(1 - \cos \theta) + R_y \sin \theta \\ \sin \theta & \cos \theta & R_y(1 - \cos \theta) - R_x \sin \theta \\ 0 & 0 & 1 \end{pmatrix} \mathbf{P}$$



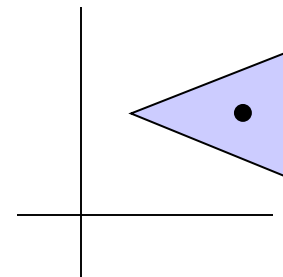
1)



2)



3)



EX.

Rotate a triangle A(0,0), B(2,2), C(4,2) about the origin and about P(-2,-2) by and angle of 45°.

Sol-

The given triangle ABC can be represented by a matrix, formed from the homogeneous coordinates of the vertices.

$$\begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$
$$R_{45^{\circ}} = \begin{bmatrix} \cos 45^{\circ} & -\sin 45^{\circ} & 0 \\ \sin 45^{\circ} & \cos 45^{\circ} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$R_{45^{\circ}} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the coordinates of the rotated triangle ABC are

$$R_{45^\circ}[ABC] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_{45^\circ}[ABC] = \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}$$

Rotate about P(-2,-2)

The rotation matrix is given by

$$R_{45^\circ}.P = T_V.R_{45^\circ}.T_{-V}$$

$$R_{45^\circ}.P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{45^\circ}.P = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -2 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\sqrt{2}-2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A'B'C') = R_{45^\circ}.P.(ABC)$$

$$(A'B'C') = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -2 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\sqrt{2}-2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A'B'C') = \begin{bmatrix} -2 & -2 & \sqrt{2}-2 \\ 2\sqrt{2}-2 & 4\sqrt{2}-2 & 5\sqrt{2}-2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A' = (-2, 2\sqrt{2}-2), B'(-2, 4\sqrt{2}-2), C'(\sqrt{2}-2, 5\sqrt{2}-2)$$

EX.

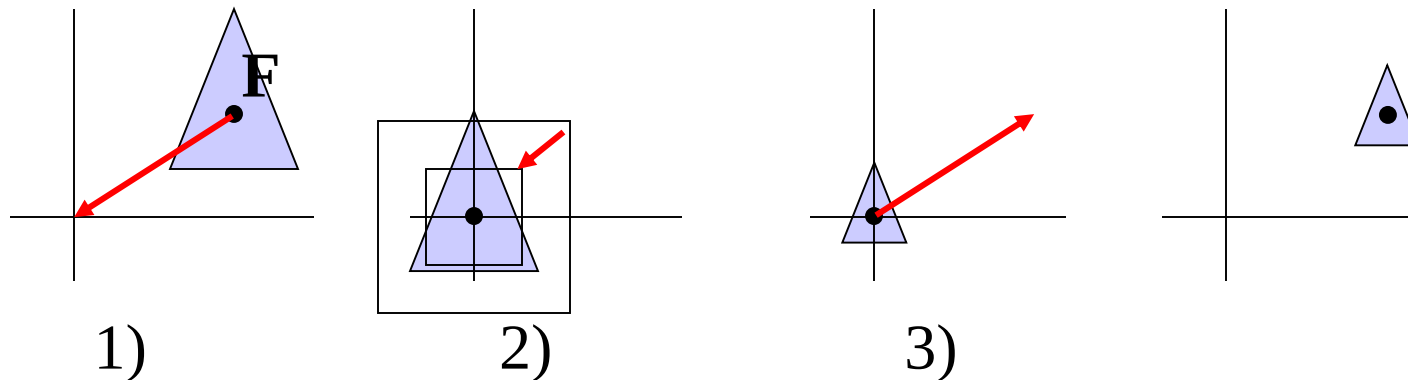
Given a triangle A(10,10), B(20,10), C(15,20) find transformed coordinates of A,B and C when the triangle is rotated in anticlockwise direction to an angle 45^0 about the fixed point (15,10).

$$(A'B'C') = \begin{bmatrix} -\frac{5}{\sqrt{2}} + 15 & \frac{5}{\sqrt{2}} + 15 & -\frac{10}{\sqrt{2}} + 15 \\ -\frac{5}{\sqrt{2}} + 10 & \frac{5}{\sqrt{2}} + 10 & \frac{10}{\sqrt{2}} + 10 \\ 1 & 1 & 1 \end{bmatrix}$$

Scaling w.r.t.(with reference to) one point

Scale with factors s_x and s_y w.r.t. point **F** :

- 1) Translate such that **F** coincides with origin;
- 2) Scaling w.r.t. origin;
- 3) Translate back again.



Scaling w.r.t.(with reference to) one point

1. Translation: $T(-F_x, -F_y)$
2. Scale: $S(S_x, S_y)$
3. Translation : $T(F_x, F_y)$

$$\begin{bmatrix} 1 & 0 & F_x \\ 0 & 1 & F_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & F_x \\ 0 & 1 & F_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & -F_x.S_x + F_x \\ 0 & S_y & -F_y.S_y + F_y \\ 0 & 0 & 1 \end{bmatrix}$$

Ex.

Find the transformation matrix that transforms the given square ABCD to half its size with centre still ramining at the same position. The coordinates of the square are: A(1,1), B(3,1). C(3,3), D(1,3) and centre at (2,2). Also find the resultant coordinates of square.

Sol-

This transformation can be carried out in the following steps:

- 1. Translate the square so that its center coincides with the origin.
- 2. Scale the squre with respect to the origin.
- 3. Translate the square back to its original position.

Thus the overall transformation matrix is formed by multiplication of three matrices.

$$[A' \quad B' \quad C' \quad D'] = \begin{bmatrix} 0.5 & 0 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 & 2.5 & 2.5 & 1.5 \\ 1.5 & 1.5 & 2.5 & 2.5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$